



NE 112 *Linear algebra for nanotechnology engineering*

1.6.9 The complex numbers are a field

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Introduction

- In this topic, we will
 - Observe that complex numbers are a field
 - Emphasize that most of the operations you performed in secondary school for reals also apply to complex numbers





Review

- In the last sequence of topics, we have defined:
 - Rectangular and polar representations
 - The complex plane
 - Absolute value or magnitude and the angle or argument
 - Addition, subtraction, the additive identity ($0 + 0j$), and additive inverses
 - Multiplication, division, the multiplicative identity ($1 + 0j$), and multiplicative inverses
 - Integer powers, complex conjugates





The complex numbers are a field

- Apart from some of the definitions, most of the *behaviors* were the same as for real numbers
 - That is, all eight axioms for a field are satisfied by complex numbers
 - Thus, what you did in secondary school, you can do here, too





The complex numbers are a field

- If z and w are complex numbers, then

$$(z+1)(z-1) = z^2 - 1$$

$$z^2 - z = z(z-1)$$

$$\frac{z^2 - 1}{z - 1} = z + 1$$

$$(w+z)(w+z) = w^2 + 2wz + z^2$$

$$\frac{1}{\frac{1}{z} + \frac{1}{w}} = \frac{zw}{z+w}$$

$$z^2 + 1 = (z-j)(z+j)$$

$$(z-j)(z-3+j)(z+4-2j) = z^3 + (1-2j)z^2 + (-11+9j)z + 10(1+j)$$





Linear ordering

- There is only one useful feature of the reals that is lost with complex numbers:
 - The reals are linearly ordered:
 - Given two real numbers x and y , then exactly one of these statements is true:
$$x < y, \ x = y, \text{ or } x > y$$
- There is no *natural* or useful linear order on complex numbers





Geometric series

- Remember those formulas for geometric series?

$$\sum_{k=0}^n z^k = \frac{1 - z^{n+1}}{1 - z}$$

- And if $|z| < 1$ then

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}$$

- Both are still true for complex numbers

$$\begin{aligned} \sum_{k=0}^5 (1 - 2j)^k &= \frac{1 - (1 - 2j)^6}{1 - (1 - 2j)} \\ &= \frac{1 - (117 - 44j)}{2j} \cdot \frac{-2j}{-2j} \\ &= \frac{88 + 232j}{4} = 22 + 58j \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{\infty} (0.5 + 0.5j)^k &= \frac{1}{1 - (0.5 + 0.5j)} \\ &= \frac{1}{0.5 - 0.5j} \cdot \frac{0.5 + 0.5j}{0.5 + 0.5j} \\ &= \frac{0.5 + 0.5j}{0.5} = 1 + j \end{aligned}$$





Geometric series

- You can even check these explicitly

$$\begin{aligned}\sum_{k=0}^5 (1-2j)^k &= 1 + (1-2j) + (-3-4j) + (-11+2j) \\ &\quad + (-7+24j) + (41+38j) = 22 + 58j\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty} (0.5+0.5j)^k &\approx 1 + (0.5+0.5j) + 0.5j + (-0.25+0.25j) \\ &\quad - 0.25 + (-0.125-0.125j) - 0.125j = 0.9375 + 0.9375j\end{aligned}$$





Roots of a polynomial

- All polynomials you will consider have real coefficients
- However, what are the roots of

$$z^2 + (-1 - 4j)z + (-1 + 17j) ?$$

$$\begin{aligned} \frac{-(-1 - 4j) \pm \sqrt{(-1 - 4j)^2 - 4(-1 + 17j)}}{2} &= \frac{1 + 4j \pm \sqrt{-11 - 60j}}{2} \\ &= \frac{1 + 4j \pm (5 - 6j)}{2} \end{aligned}$$

Thus, the roots are

$$\frac{1 + 4j + 5 - 6j}{2} = 3 - j \quad \frac{1 + 4j - 5 + 6j}{2} = -2 + 5j$$

Thus, $z^2 + (-1 - 4j)z + (-1 + 17j) = (z - 3 + j)(z + 2 - 5j)$





Summary

- In this topic, we've emphasized that the complex numbers are a field—just like the reals
 - All the properties of real numbers you're used to generally apply to complex numbers
 - The words *complex* and *imaginary* are somewhat misleading...
 - The only property we lose is linear ordering





References

- [1] https://en.wikipedia.org/wiki/Complex_number
- [2] [https://en.wikipedia.org/wiki/Field_\(mathematics\)](https://en.wikipedia.org/wiki/Field_(mathematics))
- [3] https://en.wikipedia.org/wiki/Total_order





The complex numbers are a field



Acknowledgments

None so far.





The complex numbers are a field



Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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